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ORIGINAL PAPER

A new complexity result on multiobjective linear integer programming using short rational generating functions

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Abstract This paper presents a new complexity result for solving multiobjective integer programming problems. We prove that encoding the entire set of nondominated solutions of the problem in a short sum of rational functions is polynomially doable, when the dimension of the decision space is fixed. This result extends a previous result presented in De Loera et al. (INFORMS J. Comput. 21(1):39–48, 2009) in that there the number of the objective functions is assumed to be fixed whereas ours allows this number to vary.

Keywords Multiple objective optimization · Integer programming · Short rational generating functions

1 Introduction

Short rational functions were used by Barvinok [1] as a tool to develop an algorithm for counting the number of integer points inside convex polytopes, based on the previous geometrical paper by Brion [5]. The main idea is encoding those integral points in a rational function in as many variables as the dimension of the space where the body lives. Let $P \subset \mathbb{R}^n_+$ be a given convex bounded polyhedron, the integral points may be expressed in a formal sum $\mathrm{f}(P, z) = \sum_\alpha z^\alpha$ with $\alpha = (\alpha_1, \ldots, \alpha_n) \in P \cap \mathbb{Z}^n$, where $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Barvinok's aimed objective was representing that formal sum

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of monomials in the multivariate polynomial ring $\mathbb{Z}[z_1, \ldots, z_n]$, as a "short" sum of rational functions in the same variables. Actually, Barvinok presented a polynomial-time algorithm when the dimension, n, is fixed, to compute those functions. A clear example is the polytope $P = [0, N] \subset \mathbb{R}$: the long expression of the generating function of the integer points inside P is $\mathrm{f}(P, z) = \sum_{i=0}^{N} z^i$, and it is easy to see that its representation as sum of rational functions is the well known formula $\frac{1-z^{N+1}}{1-z}$.

The above approach, apart from counting lattice points, has been used to develop some algorithms to solve, exactly, integer programming problems. Actually, De Loera et al. [8], De Loera et al. [9], and Woods and Yoshida [21] presented different methods to solve this family of problems using Barvinok's rational function of the polytope defined by the constraints of the given problem. In Lasserre [14], the author relates an integer programming problem with its linear programming relaxation and characterizes its optimal value, by using generating functions.

The goal of this paper is to present new methods for solving multiobjective integer programming problems. In contrast to usual integer programming problems, in multiobjective problems there are several (more than one) objective functions to be optimized simultaneously.

The importance of multiobjective optimization is not only due to its theoretical implications but also to its many applications. Witnesses of that are the large number of real-world decision problems that appear in the literature formulated as multiobjective programs. Examples of them are analysis in finance [11,16], vehicle routing problems [18], scheduling [15], trip organization [19], location problems [12,13] and others ([7,17]).

Multiobjective programs are formulated as optimization (we restrict ourselves without loss of generality to the maximization case) problems over feasible regions with at least two objective functions. Usually, it is not possible to maximize all the objective functions simultaneously since objective functions induce a partial order over the vectors in the feasible region, so a different notion of solution is needed. A feasible vector is said to be a *nondominated* (or *Pareto optimal*) solution if no other feasible vector has componentwise larger objective values. The evaluation through the objectives of a nondominated solution is called efficient solution.

This paper studies multiobjective integer linear programs (MOILP). Thus, we assume that there are at least two objective functions involved, the constraints that define the feasible region are linear, and the feasible vectors are integers.

Even if we assume that the objective functions are also linear, there are nowadays relatively few exact methods to solve general multiobjective integer and linear problems (see [11,23]). Some of them, as branch and bound with bound sets, which belong to the class of implicit enumeration methods, combine optimality of the returned solutions with adaptability to a wide range of problems (see for example [22] for details). Apart from dynamic programming, a different approach, as the two-phase method (see [20]), looks for supported solutions (those that can be found as solutions of a single-objective problem over the same feasible region but with objective function a linear combination of the original objectives) in a first stage and non-supported solutions are found in a second phase using the supported ones.

Nowadays, new approaches for solving multiobjective problems, using tools from algebraic geometry and computational algebra, have been proposed in the literature aiming to provide new insights into the combinatorial structure of the problems. This new research line seems to be prolific in a near future. An example of that is presented in [3] where a notion of partial Gröbner basis is given that allows to build a test family (analogous to the test set concept but for solving multiobjective problems) to solve general multiobjective linear integer programming problems. Also, in [4] the authors propose new methodologies to solve multiobjective polynomial integer programs by solving systems of polynomial equations using lexicographical Gröbner bases. Another witness of this trend is the recent work by De Loera et al. [10]. In that paper, the authors present several algorithms for MOILP using generating functions that require to fix the dimension of the decision and the objective spaces to prove polynomiality.

In this paper, we also use rational generating functions of the integer points inside rational polytopes for solving MOILP. Our main contribution is to improve the polynomiality results in [10] not requiring the dimension of the objective space to be fixed. Section 2 presents the multiobjective integer problem and the notion of dominance in order to clarify which kind of solutions we are looking for. In Sect. 3, fixing the dimension of the decision space, a polynomial time algorithm that encodes the set of nondominated solutions of the problem as a short sum of rational functions is detailed. Next, we extend the polynomial-delay polynomial-space algorithm for enumerating the solutions of a multiobjective problem presented in [10] by using our new result. The paper finishes with a concluding remark on a polynomiality result for the optimization of linear functions over the nondominated solution set of any MOILP.

2 Multiobjective integer programming

In this section we present the problem that we deal with and recall the concept of nondominated solution which is considered the standard solution set for this type of problems.

A multiobjective integer linear program (MOILP) can be formulated as:

$$\begin{aligned} \max\ & C\,x = (c_1\,x, \ldots, c_k\,x) \\ s.t.\ & \\ & \sum_{j=1}^{n} a_{ij}\,x_j \leqslant b_i,\ i = 1, \ldots, m, \\ & x_j \in \mathbb{Z}_+,\ j = 1, \ldots, n \end{aligned} \tag{$\mathrm{MOIP}_{A,C}(b)$}$$

with a_{ij}, b_i integers and $C = (c_{ij}) \in \mathbb{Z}^{k\times n}$. We will assume that the constraints define a rational polytope in $\mathbb{R}^n$. Therefore, from now on we deal with ($\mathrm{MOIP}_{A,C}(b)$).

It is clear that ($\mathrm{MOIP}_{A,C}(b)$) is not a standard optimization problem since the objective function is a k-coordinate vector, thus inducing a partial order among its feasible solutions. Hence, solving the above problem requires an alternative concept of solution, namely the set of nondominated (or Pareto optimal) points.

A vector $\widehat{x} \in \mathbb{R}^n$ is said to be a *nondominated* (or Pareto optimal) solution of ($\text{MOIP}_{A,C}(b)$) if there is no other feasible vector y such that

$$c_j\, y \geqslant c_j\, \widehat{x} \qquad \forall j = 1, \ldots, k$$

with at least one strict inequality for some j. If x is a nondominated solution, the vector $Cx = (c_1\, x, \ldots, c_k\, x) \in \mathbb{R}^k$ is called *efficient*.

We say that a dominated point, y, is dominated by x if $c_i\, x \gneqq c_i\, y$ for all $i = 1, \ldots, k$ (we denote by $\gneqq$ the binary relation "greater than or equal to" and where it is assumed that at least one of the inequalities in the list is strict).

Through this paper, we are looking for the entire set of nondominated solutions, equivalently the maximal complete set for ($\text{MOIP}_{A,C}(b)$).

3 A short rational function expression of the entire set of nondominated solutions

First of all, we recall some results on short rational functions for polytopes, that we use in our development. For details the interested reader is referred to [1,2].

Let $P = \{x \in \mathbb{R}^n : A\, x \leq b, x \geq 0\}$ be a rational polytope in $\mathbb{R}^n$. The main idea of Barvinok's Theory was encoding the integer points inside a rational polytope in a "long" sum of monomials:

$$\mathrm{f}(P; z) = \sum_{\alpha \in P \cap \mathbb{Z}^n} z^{\alpha}$$

where $z^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. Then, to re-encode, in polynomial-time for fixed dimension, these integer points in a "short" sum of rational functions in the form

$$\mathrm{f}(P; z) = \sum_{i \in I} \varepsilon_i \frac{z^{u_i}}{\prod_{j=1}^{n} (1 - z^{v_{ij}})}$$

where I is a polynomial-size indexing set, and where $\varepsilon_i \in \{1, -1\}$ and $u_i, v_{ij} \in \mathbb{Z}^n$ for all i and j (Theorem 5.4 in [2]).

It is well-known that enumerating the entire set of nondominated solutions of general multiobjective integer linear problems is #P-hard [6,11] even in fixed dimension. Therefore listing these solutions, in general, is hopeless. Nevertheless, one can try to represent these sets in polynomial time using a different strategy. Recently, De Loera et al. [10] proved that using short generating functions of rational polytopes one can encode the whole set of nondominated solutions of MOILP fixing the dimension of the space of variables and objectives. Our main contribution in this note is to extend their result allowing the number of objectives to be variable.

Theorem 1 *Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $C = (c_1, \ldots, c_k) \in \mathbb{Z}^{k \times n}$, $J \in \{1, \ldots, n\}$, and assume that the number of variables n is fixed. Suppose $P = \{x \in \mathbb{R}^n : A\, x \leq b,$*

$x \geq 0\}$ *is a rational polytope in* $\mathbb{R}^n$. *Then, we can encode, in polynomial time, the entire set of nondominated solutions for* (MOIP$_{A,C}(b)$) *in a short sum of rational functions.*

Proof Let $P_C = \{(u, v) \in \mathbb{R}^n \times \mathbb{R}^n : u, v \in P, c_i\, u - c_i\, v \geq 0$ for all $i = 1, \ldots, k$, $\sum_{i=1}^{k} c_i\, u - \sum_{i=1}^{k} c_i\, v \geq 1$, and $u, v \geq 0\}$. P_C is clearly a rational polytope. For fixed $u \in \mathbb{Z}^n$ and any $\alpha \in \mathbb{Z}^n$, $(u, \alpha) \in P_C \cap \mathbb{Z}^{2n}$ represents that α is a feasible solution dominated by u.

Now, for any rational polytope in $2n$ variables, Q, let $\pi_{1,Q}, \pi_{2,Q}$ be the short generating functions of the projections of the integer points in Q onto the first and last n coordinates, respectively. Thus $\pi_{2,P_C}(y)$ encodes all dominated feasible integral vectors of P_C and it can be computed in polynomial time by Theorem 1.7 in [2].

Furthermore, let $\mathrm{F}(x)$ be the short form of the generating function encoding the integer points in P. Both, π_{2,P_C} and $\mathrm{F}(x)$ are computed in polynomial time by Theorem 1.7 and Theorem 5.4 in [2] respectively. Compute the following difference:

$$\mathrm{h}(x) := \mathrm{F}(x) - \pi_{2,P_C}(x).$$

This is the sum over all monomials x^u where $u \in P \cap \mathbb{Z}^n$ is a nondominated solution, since we are deleting, from the total sum of feasible solutions, the set of dominated ones.

This construction gives us a short sum of rational functions associated with the sum over all monomials with degrees being the nondominated solutions of (MOIP$_{A,C}(b)$). (As a consequence, we can compute the number of nondominated solutions for the problem). The complexity of the entire construction being polynomial since we only use polynomial time operations among two generating functions of lattice points insides rational polytopes (these operations are the computation of the short rational functions $\mathrm{F}(x)$ and $\pi_{2,P_C}(x)$). □

The combination of Theorem 1 above and 7 in [10] results in the following consequence.

Theorem 2 *Assume n is a constant. There exists a polynomial-delay polynomial -space procedure to enumerate the entire set of nondominated solutions of* (MOIP$_{A,C}(b)$).

The application of the above result to the single criterion case provides an alternative proof of polynomiality for the problem of finding an optimal solution of integer linear problems, in fixed dimension. In addition, by applying Theorem 1 we can also give another polynomiality result for the optimization over the nondominated solution set of MOILP in fixed dimension. In practice, a decision maker expects to be helped by the solutions of the multiobjective problem. However, in many cases the set of nondominated solutions is too large to make easily the decision, so it is necessary to optimize (using a new criterion) over this set.

With our approach, we are able to compute, in polynomial time for fixed dimension, a “short sum of rational functions”-representation, $\mathrm{F}(z)$, of the set of nondominated solutions of ($\text{MOIP}_{A,C}(b)$). Then, using this representation the results in [10] imply that the optimization of linear functions over the efficient region of a multiobjective problem ($\text{MOIP}_{A,C}(b)$) is doable in polynomial time, fixing only the dimension of the decision space (but not the dimension of the space of objectives). The same argument also ensures the existence of a fully polynomial-time approximation scheme (FPTAS) for the minimization of non-polyhedral distance functions over the same set (in fixed dimension).

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